Cauchy Integral Formula

Ref: Complex Variables by James Ward Brown and Ruel V. Churchil

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Theorem (Cauchy Integral Formula)

Let **f** be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}. \qquad \cdots\cdots \textbf{(1)}$$

Remarks

1. This formula tells that if a function **f** is analytic within and on a simple closed contour C, then the values of **f** interior to C are completely determined by the values of **f** on C.
2. $\int_C \frac{f(z)dz}{z - z_0} = 2\pi i\, f(z_0)$ can be used to evaluate certain integrals along simple closed contours.

Example : Let C be the positively oriented circle $|Z| = 2$.

Find $\int_C \frac{zdz}{(9 - z^2)(z+i)}$

Solution :

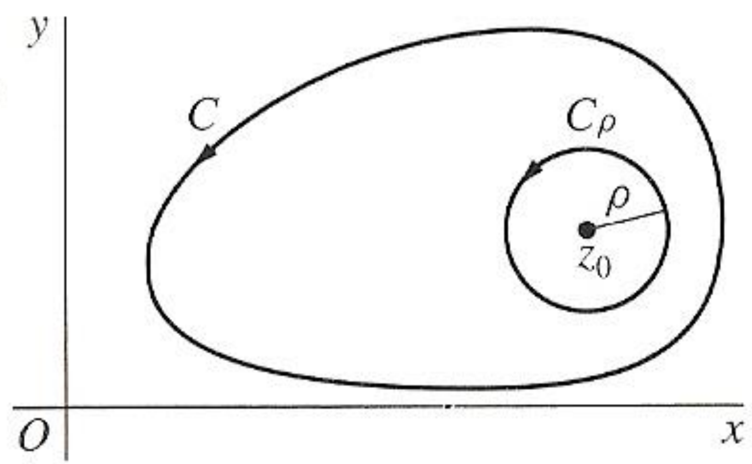
The function $f(z) = \frac{z}{(9 - z^2)}$ is analytic within and on C.

Since the point $z_0 = -i$ is interior to C, we get

$$\int_C \frac{zdz}{(9 - z^2)(z+i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)} dz = 2\pi i \, f(-i) \qquad = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Proof of the theorem :

Let C_ρ denote **a** positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C.



FIGURE

Note that the function $f(z)/(z-z_o)$ is analytic between and on the contours C and C_ρ : Then from the principle of deformation of paths it follows that

$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_\rho} \frac{f(z)dz}{z-z_0}$$

Now $\displaystyle \int_C \frac{f(z)dz}{z-z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)dz}{z-z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z-z_0}$

$$= \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \qquad \ldots\ldots \mathbf{(3)}$$

To Find $\int_{C_\rho} \frac{dz}{z - z_0}$

C_ρ is $|z - z_0| = \rho \quad \Rightarrow z - z_0 = \rho e^{i\theta} \quad (0 \le \theta \le 2\pi)$

So $\quad dz = \rho i e^{i\theta} d\theta$

Now $$\int_{C_\rho} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = 2\pi i$$

So **(3)** becomes

$$\int_C \frac{f(z)dz}{z - z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \qquad \cdots\cdots \textbf{(4)}$$

Now **f** is analytic at $z_0 \Rightarrow f$ is continuous at z_0.

So for each $\in > 0$, however small, $\exists \delta > 0$ such that

$|f(z) - f(z_0)| < \in$ whenever $|z - z_0| < \delta$ ...... **(5)**

Let $\rho < \delta$, $\Rightarrow |z - z_0| = \rho < \delta \Rightarrow |f(z) - f(z_0)| < \in$

We know $\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq ML$

So $$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z - z_0}\, dz\right| < \frac{\in}{\rho}\, 2\pi\rho = 2\pi \in$$

$$\Rightarrow \left|\int_C \frac{f(z)dz}{z - z_0} - 2\pi i\, f(z_0)\right| < 2\pi \in$$ [by **(4)]**

Since this is true for every $\in > 0$, this is also true for every $2\pi \in > 0$.

$$\Rightarrow \int_C \frac{f(z)dz}{z - z_0} = 2\pi i\, f(z_0).$$ ■